H FINITE GEOMETRIC SERIES

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SUM OF A FINITE GEOMETRIC SERIES

If the first term is u_1 and the common ratio is r, then the terms are: $u_1, u_1r, u_1r^2, u_1r^3, \ldots.., u_1r^{n-1}$.

So, $S_n = u_1 + u_1r + u_1r^2 + u_1r^3 + \ldots. + u_1r^{n-2} + u_1r^{n-1}$

For a finite geometric series with $\boldsymbol{r \neq 1}$,

$$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \qquad or \qquad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

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$$\boldsymbol{S_n = \frac{u_1(r^n - 1)}{r - 1}} \quad or \quad \boldsymbol{S_n = \frac{u_1(1 - r^n)}{1 - r}}.$$

Proof:

$$\text{If} \quad S_n = u_1 + u_1 r + u_1 r^2 + u_1 r^3 + ... + u_1 r^{n-2} + u_1 r^{n-1} \qquad (*)$$

$$\text{then} \quad rS_n = (u_1 r + u_1 r^2 + u_1 r^3 + u_1 r^4 + ... + u_1 r^{n-1}) + u_1 r^n$$

$$\therefore \quad rS_n = (S_n - u_1) + u_1 r^n \qquad \{\text{from } (*)\}$$

$$\therefore \quad rS_n - S_n = u_1 r^n - u_1$$

$$\therefore \quad S_n(r - 1) = u_1(r^n - 1)$$

$$\therefore \quad S_n = \frac{u_1(r^n - 1)}{r - 1} \quad \text{or} \quad \frac{u_1(1 - r^n)}{1 - r} \quad \text{provided} \quad r \neq 1.$$

In the case $r = 1$ we have a sequence in which all terms are the same. The sequence is also arithmetic (with $d = 0$), and $S_n = u_1 n$.

Example 28 **Self Tutor**

Find the sum of $2 + 6 + 18 + 54 + \ldots.$ to 12 terms.

Example 28 **Self Tutor**

Find the sum of $2+6+18+54+....$ to 12 terms.

The series is geometric with $u_1 = 2$, $r = 3$, and $n = 12$.

$$S_n = \frac{u_1(r^n - 1)}{r-1}$$

$$\therefore \quad S_{12} = \frac{2(3^{12}-1)}{3-1} = 531\,440$$

Example 29

Self Tutor

Find a formula for S_n, the sum of the first n terms of the series $9 - 3 + 1 - \frac{1}{3} + ...$

This answer cannot be simplified as we do not know if n is odd or even.

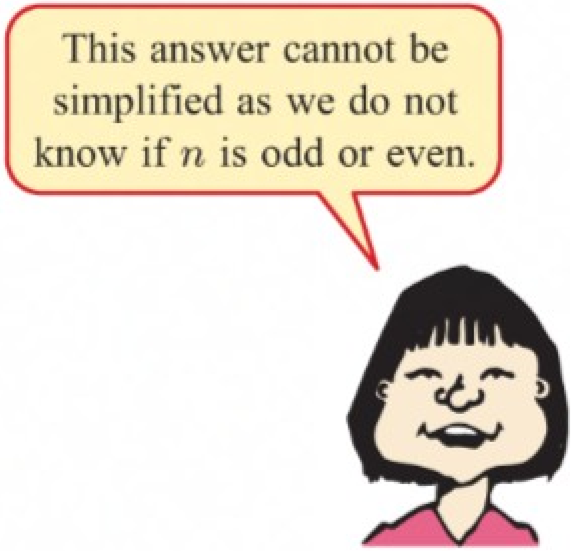
Example 29 **Self Tutor**

Find a formula for S_n, the sum of the first n terms of the series $9-3+1-\frac{1}{3}+....$

The series is geometric with $u_1=9$ and $r=-\frac{1}{3}$.

$$S_n=\frac{u_1(1-r^n)}{1-r}=\frac{9(1-(-\frac{1}{3})^n)}{\frac{4}{3}}$$

$$\therefore\quad S_n=\frac{27}{4}(1-(-\frac{1}{3})^n)$$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1 Find the sum of the following series:

- **a** $2 + 6 + 18 + 54 + ...$ to 8 terms
- **b** $5 + 10 + 20 + 40 + ...$ to 10 terms
- **c** $12 + 6 + 3 + 1.5 + ...$ to 10 terms
- **d** $\sqrt{7} + 7 + 7\sqrt{7} + 49 + ...$ to 12 terms
- **e** $6 - 3 + 1\frac{1}{2} - \frac{3}{4} + ...$ to 15 terms
- **f** $1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{2\sqrt{2}} + ...$ to 20 terms.

2 Find a formula for S_n, the sum of the first n terms of the series:

- **a** $\sqrt{3} + 3 + 3\sqrt{3} + 9 + ...$
- **b** $12 + 6 + 3 + 1\frac{1}{2} + ...$
- **c** $0.9 + 0.09 + 0.009 + 0.0009 + ...$
- **d** $20 - 10 + 5 - 2\frac{1}{2} + ...$

3 Evaluate these geometric series:

- **a** $\sum_{k=1}^{10} 3 \times 2^{k-1}$
- **b** $\sum_{k=1}^{12} \left(\frac{1}{2}\right)^{k-2}$
- **c** $\sum_{k=1}^{25} 6 \times (-2)^k$

4 At the end of each year, a salesperson is paid a bonus of \$2000 which is always deposited into the same account. It earns a fixed rate of interest of 6% p.a. with interest being paid annually. The total amount in the account at the end of each year will be:

$$\begin{aligned} A_1 &= 2000 \\ A_2 &= A_1 \times 1.06 + 2000 \\ A_3 &= A_2 \times 1.06 + 2000 \quad \text{and so on.} \end{aligned}$$

- **a** Show that $A_3 = 2000 + 2000 \times 1.06 + 2000 \times (1.06)^2$.
- **b** Show that $A_4 = 2000[1 + 1.06 + (1.06)^2 + (1.06)^3]$.
- **c** Find the total bank balance after 10 years, assuming there are no fees or withdrawals.

5 Answer the **Opening Problem** on page **90**.

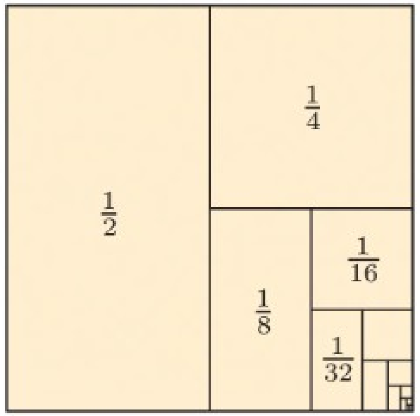
6 Paula has started renting an apartment. She paid \$5000 rent in the first year, and the rent increased by 5% each year.

- **a** Find, to the nearest \$10, the rent paid by Paula in the 4th year.
- **b** Write an expression for the total rent paid by Paula during the first n years.
- **c** How much rent did Paula pay during the first 7 years? Give your answer to the nearest \$10.

7 Jim initially deposits £6000 in an account which earns 5% p.a. interest paid annually. At the end of each year, Jim invests another £1000 in the account. Find the value of the account after 8 years.

8 Consider $S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^n}$.

- **a** Find S_1, S_2, S_3, S_4, and S_5 in fractional form.
- **b** Hence guess the formula for S_n.
- **c** Find S_n using $S_n = \dfrac{u_1(1-r^n)}{1-r}$.
- **d** Comment on S_n as n gets very large.
- **e** Explain the relationship between the given diagram and **d**.



9 A geometric series has second term 6. The sum of its first three terms is -14. Find its fourth term.

10 An arithmetic and a geometric sequence both have first term 1, and their second terms are equal. The 14th term of the arithmetic sequence is three times the third term of the geometric sequence. Find the twentieth term of each sequence.

11 Suppose $u_1, u_2, \dots, u_n$ is a geometric sequence with common ratio r. Show that

$$(u_1+u_2)^2 + (u_2+u_3)^2 + (u_3+u_4)^2 + \dots + (u_{n-1}+u_n)^2 = \frac{2{u_1}^2(r^{2n-1}-1)}{r-1} - ({u_1}^2 + {u_n}^2).$$

Example 30 **Self Tutor**

A geometric sequence has first term 5 and common ratio 2. The sum of the first n terms of the sequence is 635. Find n.

Example 30

Self Tutor

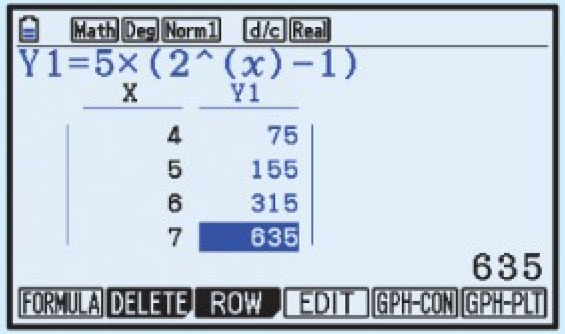
A geometric sequence has first term 5 and common ratio 2. The sum of the first n terms of the sequence is 635. Find n.

The sequence is geometric with $u_1 = 5$ and $r = 2$.

$$\therefore \quad S_n = \frac{u_1(r^n - 1)}{r - 1} = \frac{5(2^n - 1)}{2 - 1} = 5(2^n - 1)$$

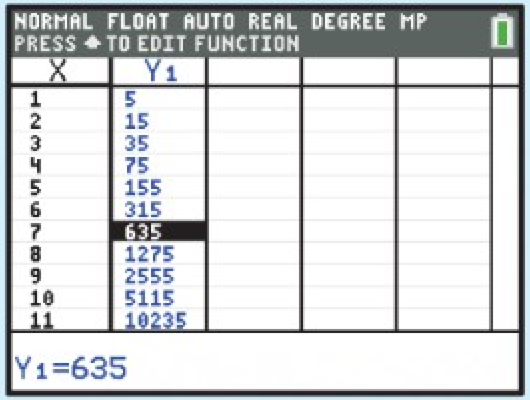
To find n such that $S_n = 635$, we use a table of values with $\text{Y}_1 = 5 \times (2\,\hat{}\,\text{X} - 1)$:

Casio fx-CG50



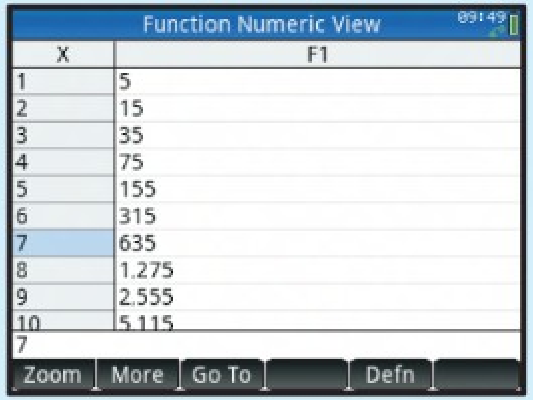
```
Y1=5×(2^(x)−1)
      X        Y1
      4        75
      5       155
      6       315
      7       635
                    635
```

TI-84 Plus CE



```
NORMAL FLOAT AUTO REAL DEGREE MP
PRESS + TO EDIT FUNCTION
 X     Y₁
 1     5
 2     15
 3     35
 4     75
 5     155
 6     315
 7     635
 8     1275
 9     2555
 10    5115
 11    10235
Y₁=635
```

HP Prime



```
 X     F1
 1     5
 2     15
 3     35
 4     75
 5     155
 6     315
 7     635
 8     1.275
 9     2.555
 10    5.115
7
```

$S_7 = 635$, so $n = 7$.

12 A geometric sequence has first term 6 and common ratio 1.5. The sum of the first n terms of the sequence is 79.125. Find n.

13 Find n given that $\sum_{k=1}^{n} 2 \times 3^{k-1} = 177\,146$.

14 Felicity is offered a new job, and is given two salary options to choose from:

Option A: $40 000 in the first year, and 5% extra each subsequent year.

Option B: $60 000 in the first year, and $1000 more each subsequent year.

a If Felicity believed that she would work for 3 years in this new job, explain why *Option B* would be best for her.

b Write down an expression for the amount of money earned in the nth year if she selects:

i *Option A* **ii** *Option B*.

c Find the minimum length of time Felicity would need to work before the amount of money earned per year from *Option A* exceeds that of *Option B*.

d Felicity decides that the best way to compare the two options is to consider the *total* income accumulated after the first n years in each case. If T_A and T_B represent the total income earned over n years for *Options A* and *B* respectively, show that:

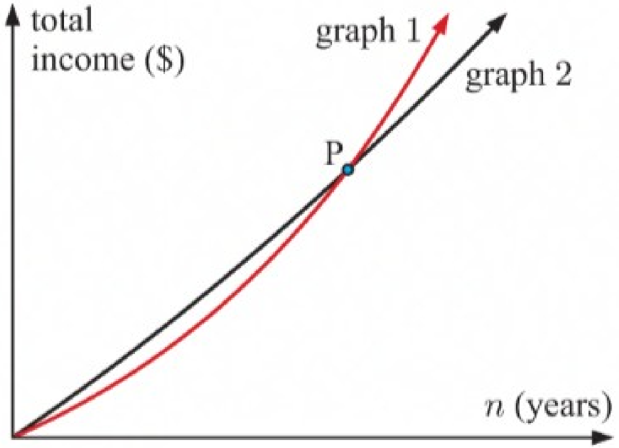
i $T_A = 800\,000(1.05^n - 1)$ dollars **ii** $T_B = 500n^2 + 59\,500n$ dollars

e The graph alongside shows T_A and T_B graphed against n.

i Which graph represents T_A and which graph represents T_B?

ii Use technology to find the coordinates of the point P, where T_A and T_B intersect.

iii Hence write down a time interval, in whole years, for which *Option B* provides the greater total income.



15 \$8000 is borrowed over a 2-year period at a rate of 12% p.a. compounded quarterly. Quarterly repayments are made and the interest is adjusted each quarter, which means that at the end of each quarter, interest is charged on the previous balance and then the balance is reduced by the amount repaid.

There are $2 \times 4 = 8$ repayments and the interest per quarter is $\dfrac{12\%}{4} = 3\%$.

At the end of the first quarter, the amount owed is given by $A_1 = \$8000 \times 1.03 - R$, where R is the amount of each repayment.

At the end of the second quarter, the amount owed is given by:

$$\begin{aligned} A_2 &= A_1 \times 1.03 - R \\ &= (\$8000 \times 1.03 - R) \times 1.03 - R \\ &= \$8000 \times (1.03)^2 - 1.03R - R \end{aligned}$$

- **a** Write an expression for the amount owed at the end of the third quarter, A_3.
- **b** Write an expression for the amount owed at the end of the eighth quarter, A_8.
- **c** Given that $A_8 = 0$ for the loan to be fully repaid, deduce the value of R.
- **d** Now suppose the amount borrowed was $\$P$, r is the interest rate per repayment interval (as a decimal), and there are m repayments. Show that each repayment is

  $R = \dfrac{P(1+r)^m \times r}{(1+r)^m - 1}$ dollars.

1 **a** 6560 **b** 5115 **c** $\frac{3069}{128} \approx 24.0$

d $\approx 189\,000$ **e** $\frac{32\,769}{8192} \approx 4.00$ **f** ≈ 0.585

2 **a** $S_n = \dfrac{3+\sqrt{3}}{2}\left((\sqrt{3})^n - 1\right)$ **b** $S_n = 24\left(1 - (\tfrac{1}{2})^n\right)$

c $S_n = 1 - (0.1)^n$ **d** $S_n = \frac{40}{3}\left(1 - (-\tfrac{1}{2})^n\right)$

3 **a** 3069 **b** $\frac{4095}{1024} \approx 4.00$ **c** $-134\,217\,732$

4 **c** \$26 361.59

5 **a** The number of grains of wheat starts at 1, and each square has double the number of grains of the previous square.

c $u_n = 2^{n-1}$ **d** $(2^{64} - 1) \approx 1.84 \times 10^{19}$ grains of wheat

6 **a** \$5790 **b** $S_n = 100\,000((1.05)^n - 1)$ **c** $\approx \$40\,710$

7 £18 413.84

8 **a** $S_1 = \frac{1}{2}$, $S_2 = \frac{3}{4}$, $S_3 = \frac{7}{8}$, $S_4 = \frac{15}{16}$, $S_5 = \frac{31}{32}$

b $S_n = \dfrac{2^n - 1}{2^n}$ **c** $S_n = 1 - (\frac{1}{2})^n = \dfrac{2^n - 1}{2^n}$

d as $n \to \infty$, $S_n \to 1$

e As $n \to \infty$, the sum of the fractions approaches the area of a 1×1 unit square.

9 $u_4 = \frac{2}{3}$ or 54

10 Arithmetic: $u_{20} = 39$ or $7\frac{1}{3}$

Geometric: $u_{20} = 3^{19}$ or $\left(\frac{4}{3}\right)^{19}$

12 $n = 5$ **13** $n = 11$

14 **a** In 3 years she will earn \$183 000 under *Option B*, compared with \$126 100 under *Option A*.

b **i** $A_n = 40\,000 \times (1.05)^{n-1}$ **ii** $B_n = 59\,000 + 1000n$

c ≈ 13.1 years

e **i** graph 1 represents T_A, graph 2 represents T_B

ii P(22.3, 1 580 000) **iii** $0 \leqslant n \leqslant 22$

15 **a** $A_3 = \$8000 \times (1.03)^3 - (1.03)^2R - 1.03R - R$

b $A_8 = \$8000 \times (1.03)^8 - (1.03)^7R - (1.03)^6R - (1.03)^5R - (1.03)^4R - (1.03)^3R - (1.03)^2R - (1.03)R - R$

c $R = \$1139.65$